

# LEXICOGRAPHIC PSEUDO MV-ALGEBRAS

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**ABSTRACT.** A lexicographic pseudo MV-algebra is an algebra that is isomorphic to an interval in the lexicographic product of a linear unital group with an arbitrary  $\ell$ -group. We present conditions when a pseudo MV-algebra is lexicographic. We show that a key condition is the existence of a lexicographic ideal, or equivalently, a case when the algebra can be split into comparable slices indexed by elements of the interval  $[0, u]$  of some unital linearly ordered group  $(H, u)$ . Finally, we show that fixing  $(H, u)$ , the category of  $(H, u)$ -lexicographic pseudo MV-algebras is categorically equivalent to the category of  $\ell$ -groups.

## 1. INTRODUCTION

MV-algebras are the algebraic counterpart of the infinite-valued Łukasiewicz sentential calculus introduced by Chang in [Cha]. Perfect MV-algebras were characterized as MV-algebras where each element is either infinitesimal or co-infinitesimal. Therefore, they have no parallels in the realm of Boolean algebras because perfect MV-algebras are not semisimple. The logic of perfect pseudo MV-algebras has a counterpart in the Lindenbaum algebra of the first order Łukasiewicz logic which is not semisimple, because the valid but unprovable formulas are precisely the formulas that correspond to co-infinitesimal elements of the Lindenbaum algebra, see e.g. [DiGr]. Therefore, the study of perfect MV-algebras is tightly connected with this important phenomenon of the first order Łukasiewicz logic.

Recently, two equivalent non-commutative generalizations of MV-algebras, called pseudo MV-algebras in [GeIo] or GMV-algebras in [Rac], were introduced. They are used for algebraic description of non-commutative fuzzy logic, see [Haj]. For them the author [Dvu2] generalized a well-known Mundici's representation theorem, see e.g. [CDM, Cor 7.1.8], showing that every pseudo MV-algebra is always an interval in a unital  $\ell$ -group not necessarily Abelian.

From algebraic point of view of perfect MV-algebras, it was shown in [DiLe1] that every perfect MV-algebra  $M$  can be represented as an interval in the lexicographic

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<sup>1</sup>Keywords: Pseudo MV-algebra,  $\ell$ -group, strong unit, lexicographic product, ideal, lexicographic ideal,  $(H, u)$ -perfect pseudo MV-algebra, strongly  $(H, u)$ -perfect pseudo MV-algebra,  $(H, u)$ -lexicographic pseudo MV-algebra, weakly  $(H, u)$ -lexicographic pseudo MV-algebra

AMS classification: 06D35, 03G12

This work was supported by the Slovak Research and Development Agency under contract APVV-0178-11, grant VEGA No. 2/0059/12 SAV, and GAČR 15-15286S.

product, i.e.  $M \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ . This result was extended also for perfect effect algebras [Dvu3].

This notion was generalized in [Dvu4] to  $n$ -perfect pseudo MV-algebras, they can be decomposed into  $(n + 1)$ -comparable slices, and they can be represented in the form  $\Gamma(\frac{1}{n}\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ .  $\mathbb{R}$ -perfect pseudo MV-algebras can be represented in the form  $\Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$ , see [Dvu5], if  $G$  is Abelian, such MV-algebras were studied in [DiLe2]. Recently, lexicographic MV-algebras were studied in [DFL], they have a representation in the form  $\Gamma(H \overrightarrow{\times} G, (u, 0))$ , where  $(H, u)$  is an Abelian linearly ordered group and  $G$  is an Abelian  $\ell$ -group.

Thus we see that MV-algebras and pseudo MV-algebras that can be represented in the form  $\Gamma(H \overrightarrow{\times} G, (u, 0))$  are intensively studied in the last period, see [Dvu6] where  $H$  was assumed to be Abelian. In this contribution, we continue in this study exhibiting the most general case of  $(H, u)$  and  $G$  when they are not assumed to be Abelian. We show that the crucial conditions are the existence of a lexicographic ideal, or equivalently, the possibility to decompose  $M$  into comparable slices indexed by the elements of the interval  $[0, u]_H$ ; we call such algebras  $(H, u)$ -perfect. In addition, we present also conditions when  $M$  can be represented in the form  $\Gamma(H \overrightarrow{\times} G, (u, b))$ , where  $b \in G^+$  is not necessarily the zero element.

The paper is organized as follows. The second section gathers the basic notions on pseudo MV-algebras. In the third section we introduce a lexicographic ideal, and we present a representation of a pseudo MV-algebra in the form  $\Gamma(H \overrightarrow{\times} G, (u, 0))$ . Section 4 gives a categorical equivalence of the category of  $(H, u)$ -lexicographic pseudo MV-algebras to the category of  $\ell$ -groups. The final section will describe weakly  $(H, u)$ -perfect pseudo MV-algebras; they can be represented in the form  $\Gamma(H \overrightarrow{\times} G, (u, b))$ , where  $b$  can be even strictly positive. Crucial notions for such algebras are a weakly lexicographic ideal as well as a weakly  $(H, u)$ -perfect pseudo MV-algebra.

## 2. BASIC NOTIONS ON PSEUDO MV-ALGEBRAS

According to [GeIo], a *pseudo MV-algebra* or a *GMV-algebra* by [Rac] is an algebra  $(M; \oplus, ^-, \sim, 0, 1)$  of type  $(2, 1, 1, 0, 0)$  such that the following axioms hold for all  $x, y, z \in M$  with an additional binary operation  $\odot$  defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

- (A1)  $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ ;
- (A2)  $x \oplus 0 = 0 \oplus x = x$ ;
- (A3)  $x \oplus 1 = 1 \oplus x = 1$ ;
- (A4)  $1^\sim = 0$ ;  $1^- = 0$ ;
- (A5)  $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$ ;
- (A6)  $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$ ;<sup>2</sup>
- (A7)  $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$ ;
- (A8)  $(x^-)^\sim = x$ .

Any pseudo MV-algebra is a distributive lattice where (A6) and (A7) define the joint  $x \vee y$  and the meet  $x \wedge y$  of  $x, y$ , respectively.

A pseudo MV-algebra  $M$  is an *MV-algebra* if  $x \oplus y = y \oplus x$  for all  $x, y \in M$ .

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<sup>2</sup> $\odot$  has a higher binding priority than  $\oplus$ .

Orthodox examples of pseudo MV-algebras are generated by unital  $\ell$ -groups not necessarily Abelian.

We note that a *po-group* (= partially ordered group) is a group  $(G; +, 0)$  (written additively) endowed with a partial order  $\leq$  such that if  $a \leq b$ ,  $a, b \in G$ , then  $x + a + y \leq x + b + y$  for all  $x, y \in G$ . We denote by  $G^+ = \{g \in G : g \geq 0\}$  the *positive cone* of  $G$ . If, in addition,  $G$  is a lattice under  $\leq$ , we call it an  $\ell$ -group (= lattice ordered group). An element  $u \in G^+$  is said to be a *strong unit* (= order unit) if  $G = \bigcup_n [-nu, nu]$ , and the couple  $(G, u)$  with a fixed strong unit  $u$  is said to be a *unital po-group* or a *unital  $\ell$ -group*, respectively. The *commutative center* of a group  $H$  is the set  $C(H) = \{h \in H : h + h' = h' + h, \forall h' \in H\}$ . We denote by  $[0, u]_H := \{h \in H : 0 \leq h \leq u\}$  for each  $u \in H^+$ .

Finally, two unital  $\ell$ -groups  $(G, u)$  and  $(H, v)$  are *isomorphic* if there is an  $\ell$ -group isomorphism  $\phi : G \rightarrow H$  such that  $\phi(u) = v$ . In a similar way an isomorphism and a homomorphism of unital po-groups are defined. For more information on po-groups and  $\ell$ -groups and for unexplained notions about them, see [Dar, Fuc, Gla].

By  $\mathbb{R}$  and  $\mathbb{Z}$  we denote the groups of reals and natural numbers, respectively.

Between pseudo MV-algebras and unital  $\ell$ -groups there is a very close connection: If  $u$  is a strong unit of a (not necessarily Abelian)  $\ell$ -group  $G$ ,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

then  $(\Gamma(G, u); \oplus, ^-, \sim, 0, u)$  is a pseudo MV-algebra.

The basic representation theorem for pseudo MV-algebras is the following generalization [Dvu2] of the Mundici famous result:

**Theorem 2.1.** *For any pseudo MV-algebra  $(M; \oplus, ^-, \sim, 0, 1)$ , there exists a unique (up to isomorphism) unital  $\ell$ -group  $(G, u)$  such that  $(M; \oplus, ^-, \sim, 0, 1)$  is isomorphic to  $(\Gamma(G, u); \oplus, ^-, \sim, 0, u)$ . The functor  $\Gamma$  defines a categorical equivalence of the category of pseudo MV-algebras with the category of unital  $\ell$ -groups.*

We recall that in the category of pseudo MV-algebras objects are pseudo MV-algebras, and morphisms are homomorphisms of pseudo MV-algebras, whereas objects in the category of unital  $\ell$ -groups are unital  $\ell$ -groups  $(G, u)$ , and morphisms are homomorphisms of  $\ell$ -groups preserving fixed strong units.

We note that the class of pseudo MV-algebras is a variety whereas the class of unital  $\ell$ -groups is not a variety because it is not closed under infinite products.

Due to this result, if  $M = \Gamma(G, u)$  for some unital  $\ell$ -group  $(G, u)$ , then  $M$  is linearly ordered iff  $G$  is a linearly ordered group, see [Dvu1, Thm 5.3].

Besides a total operation  $\oplus$ , we can define a partial operation  $+$  on any pseudo MV-algebra  $M$  in such a way that  $x + y$  is defined iff  $x \odot y = 0$  and then we set

$$x + y := x \oplus y. \tag{2.1}$$

In other words,  $x + y$  is precisely the group addition  $x + y$  if the group sum  $x + y$  is defined in  $M$ .

Let  $A, B$  be two subsets of  $M$ . We define (i)  $A \leq B$  if  $a \leq b$  for all  $a \in A$  and all  $b \in B$ , (ii)  $A \oplus B = \{a \oplus b : a \in A, b \in B\}$ , and (iii)  $A + B = \{a + b : \text{if } a + b \text{ exists in } M \text{ for } a \in A, b \in B\}$ , where the partial  $+$  is defined by (2.1). We say that  $A + B$  is *defined* in  $M$  if  $a + b$  exists in  $M$  for each  $a \in A$  and each  $b \in B$ . (iv)  $A^- = \{a^- : a \in A\}$  and  $A^\sim = \{a^\sim : a \in A\}$ .

Using Theorem 2.1, we have if  $y \leq x$ , then  $x \odot y^- = x - y$  and  $y^\sim \odot x = -y + x$ , where the subtraction  $-$  is in fact the group subtraction in the representing unital  $\ell$ -group.

Given an element  $x$  and any integer  $n \geq 0$ , we define

$$0x := 0, \quad 1x := x, \quad (n+1)x := (nx) + x,$$

if  $nx$  and  $(nx) + x$  are defined in  $M$ , where the simple  $+$  is defined by (2.1). An element  $x$  of  $M$  is (i) *infinitesimal* if  $nx$  is defined in  $M$  for each integer  $n \geq 1$ , (ii) *co-infinitesimal* if  $x^-$  is an infinitesimal. We denote by  $\text{Infin}(M)$  the set of infinitesimal elements of  $M$ .

We recall that if  $H$  and  $G$  are two po-groups, then the *lexicographic product*  $H \overrightarrow{\times} G$  is the group  $H \times G$  which is endowed with the lexicographic order:  $(h, g) \leq (h_1, g_1)$  iff  $h < h_1$  or  $h = h_1$  and  $g \leq g_1$ . The lexicographic product  $H \overrightarrow{\times} G$  with non-trivial  $G$  is an  $\ell$ -group iff  $H$  is linearly ordered group and  $G$  is an arbitrary  $\ell$ -group, [Fuc, (d) p. 26]. If  $G = O$ , the trivial group, then  $H \overrightarrow{\times} O$  is an  $\ell$ -group that is isomorphic to  $H$  for every  $\ell$ -group  $H$  (not necessarily linearly ordered). If  $u$  is a strong unit for  $H$ , then  $(u, 0)$  is a strong unit for  $H \overrightarrow{\times} G$ , and  $\Gamma(H \overrightarrow{\times} G, (u, 0))$  is a pseudo MV-algebra.

We say that a pseudo MV-algebra  $M$  is *symmetric* if  $x^- = x^\sim$  for all  $x \in M$ . A pseudo MV-algebra  $\Gamma(G, u)$  is symmetric iff  $u \in C(G)$ , and the variety of symmetric pseudo MV-algebras is a proper subvariety of the variety of pseudo MV-algebras  $\mathcal{PMV}$ . For example,  $\Gamma(\mathbb{R} \overrightarrow{\times} G, (1, 0))$  is symmetric and it is an MV-algebra iff  $G$  is Abelian.

An *ideal* of a pseudo MV-algebra  $M$  is any non-empty subset  $I$  of  $M$  such that (i)  $a \leq b \in I$  implies  $a \in I$ , and (ii) if  $a, b \in I$ , then  $a \oplus b \in I$ . An ideal  $I$  is said to be (i) *maximal* if  $I \neq M$  and it is not a proper subset of another ideal  $J \neq M$ ; we denote by  $\mathcal{M}(M)$  the set of maximal ideals of  $M$ , (ii) *prime* if  $x \wedge y \in I$  implies  $x \in I$  or  $y \in I$ , and (iii) *normal* if  $\{x\} \oplus I = I \oplus \{x\}$  for any  $x \in M$ .

If  $I$  is a subset of  $M$ , then  $\langle I \rangle$  denotes the least subalgebra of  $M$  generated by  $I$ . If  $I$  is an ideal of  $M$  such that  $I^- = I^\sim$ , then it is easy to see that  $\langle I \rangle = I \cup I^- = I \cup I^\sim$ .

There is a one-to-one correspondence between normal ideals and congruences for pseudo MV-algebras, [GeIo, Thm 3.8]. The quotient pseudo MV-algebra over a normal ideal  $I$ ,  $M/I$ , is defined as the set of all elements of the form  $x/I := \{y \in M : x \odot y^- \oplus y \odot x^- \in I\}$ , or equivalently,  $x/I := \{y \in M : x^\sim \odot y \oplus y^\sim \odot x \in I\}$ .

The notion of a state is an analogue of a probability measure for pseudo MV-algebras. We say that a mapping  $s$  from a pseudo MV-algebra  $M$  into the real interval  $[0, 1]$  is a *state* if (i)  $s(a + b) = s(a) + s(b)$  whenever  $a + b$  is defined in  $M$ , and (ii)  $s(1) = 1$ . We define the *kernel* of  $s$  as the set  $\text{Ker}(s) = \{a \in M : s(a) = 0\}$ . Then  $\text{Ker}(s)$  is a normal ideal of  $M$ .

Pseudo MV-algebras can be exhibited also in the realm of pseudo effect algebras with a special type of the Riesz Decomposition Property which are a non-commutative generalization of effect algebras introduced by [FoBe].

According to [DvVe1, DvVe2], a partial algebraic structure  $(E; +, 0, 1)$ , where  $+$  is a partial binary operation and 0 and 1 are constants, is called a *pseudo effect algebra* if, for all  $a, b, c \in E$ , the following hold:

- (PE1)  $a + b$  and  $(a + b) + c$  exist if and only if  $b + c$  and  $a + (b + c)$  exist, and in this case,  $(a + b) + c = a + (b + c)$ ;
- (PE2) there are exactly one  $d \in E$  and exactly one  $e \in E$  such that  $a + d = e + a = 1$ ;
- (PE3) if  $a + b$  exists, there are elements  $d, e \in E$  such that  $a + b = d + a = b + e$ ;
- (PE4) if  $a + 1$  or  $1 + a$  exists, then  $a = 0$ .

If we define  $a \leq b$  if and only if there exists an element  $c \in E$  such that  $a + c = b$ , then  $\leq$  is a partial ordering on  $E$  such that  $0 \leq a \leq 1$  for any  $a \in E$ . It is possible to show that  $a \leq b$  if and only if  $b = a + c = d + a$  for some  $c, d \in E$ . We write  $c = a / b$  and  $d = b \setminus a$ . Then

$$(b \setminus a) + a = a + (a / b) = b,$$

and we write  $a^- = 1 \setminus a$  and  $a^\sim = a / 1$  for any  $a \in E$ .

If  $(G, u)$  is a unital po-group, then  $(\Gamma(G, u); +, 0, u)$ , where the set  $\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$  is endowed with the restriction of the group addition  $+$  to  $\Gamma(G, u)$  and with 0 and  $u$  as 0 and 1, is a pseudo effect algebra. Due to [DvVe1, DvVe2], if a pseudo effect algebra satisfies a special type of the Riesz Decomposition Property,  $\text{RDP}_1$ , then every pseudo effect algebra is an interval in some unique (up to isomorphism of unital po-groups)  $(G, u)$  satisfying also  $\text{RDP}_1$  such that  $M \cong \Gamma(G, u)$ .

We say that a mapping  $f$  from one pseudo effect algebra  $E$  onto a second one  $F$  is a *homomorphism* (of pseudo effect algebras) if (i)  $a, b \in E$  such that  $a + b$  is defined in  $E$ , then  $f(a) + f(b)$  is defined in  $F$  and  $f(a + b) = f(a) + f(b)$ , and (ii)  $f(1) = 1$ . Clearly, every homomorphism of pseudo effect algebras preserves  $-$  and  $\sim$ . A bijective mapping  $h : E \rightarrow F$  is an *isomorphism* if both  $h$  and  $h^{-1}$  are homomorphisms of pseudo effect algebras.

We say that a pseudo effect algebra  $E$  satisfies  $\text{RDP}_2$  property if  $a_1 + a_2 = b_1 + b_2$  implies that there are four elements  $c_{11}, c_{12}, c_{21}, c_{22} \in E$  such that (i)  $a_1 = c_{11} + c_{12}$ ,  $a_2 = c_{21} + c_{22}$ ,  $b_1 = c_{11} + c_{21}$  and  $b_2 = c_{12} + c_{22}$ , and (ii)  $c_{12} \wedge c_{21} = 0$ .

In [DvVe2, Thm 8.3, 8.4], it was proved that if  $(M; \oplus, ^-, \sim, 0, 1)$  is a pseudo MV-algebra, then  $(M; +, 0, 1)$ , where  $+$  is defined by (2.1), is a pseudo effect algebra with  $\text{RDP}_2$ . Conversely, if  $(E; +, 0, 1)$  is a pseudo effect algebra with  $\text{RDP}_2$ , then  $E$  is a lattice, and by [DvVe2, Thm 8.8],  $(E; \oplus, ^-, \sim, 0, 1)$ , where

$$a \oplus b := (b^- \setminus (a \wedge b^-))^\sim, \quad a, b \in E, \quad (2.2)$$

is a pseudo MV-algebra.

### 3. LEXICOGRAPHIC IDEALS AND LEXICOGRAPHIC PSEUDO MV-ALGEBRAS

We say that a pseudo MV-algebra  $M$  is *lexicographic* if there are a linearly ordered unital group  $(H, u)$  and an  $\ell$ -group  $G$  (both groups are not necessarily Abelian) such that  $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$ . The main aim of this section is to show conditions when a pseudo MV-algebra is lexicographic. We show that such conditions are closely connected with the existence of a lexicographic ideal.

As a matter of interest, if  $O$  is the zero group, then  $\Gamma(O \overrightarrow{\times} G, (0, 0))$  is a one-element pseudo MV-algebra. The pseudo MV-algebra  $\Gamma(\mathbb{Z} \overrightarrow{\times} O, (1, 0))$  is a two-element Boolean algebra.

A normal ideal  $I$  of a pseudo MV-algebra  $M$  is said to be *retractive* if the canonical projection  $\pi_I : M \rightarrow M/I$  is retractive, i.e. there is a homomorphism  $\delta_I : M/I \rightarrow M$  such that  $\pi_I \circ \delta_I = id_{M/I}$ . If a normal ideal  $I$  is retractive, then  $\delta_I$  is injective and  $M/I$  is isomorphic to a subalgebra of  $M$ .

For example, if  $M = \Gamma(H \overrightarrow{\times} G, (u, 0))$  and  $I = \{(0, g) : g \in G^+\}$ , then  $I$  is a normal ideal, and due to  $M/I \cong \Gamma(H, u) \cong \Gamma(H \overrightarrow{\times} \{0\}, (u, 0)) \subseteq \Gamma(H \overrightarrow{\times} G, (u, 0))$ . If we set  $\delta_I((h, g)/I) := (h, 0)$ , we see that  $I$  is retractive.

We say that a normal ideal  $I$  of a pseudo MV-algebra  $M$  is *strict* if  $x/I < y/I$  implies  $x < y$ .

**Definition 3.1.** A normal ideal  $I$  of a pseudo MV-algebra  $M = \Gamma(G, u)$ ,  $\{0\} \neq I \neq M$ , is said to be *lexicographic* if

- (i)  $I$  is strict;
- (ii)  $I$  is retractive;
- (iii)  $I$  is prime;
- (iv) for each  $s, t \in [0, u]_H$ , where  $\Gamma(H, u) := M/I$ , such that  $s + t \leq u$  and for each  $x \in \pi_I^{-1}(\{s\})$  and  $y \in \pi_I^{-1}(\{t\})$ , we have  $x + y - \delta_I(s + t) = (x - \delta_I(s)) + (y - \delta_I(t))$ , where  $+$  and  $-$  are counted in the group  $G$ ,
- (v) for each  $t \in [0, u]_H$  and each  $x \in \pi_I^{-1}(\{t\})$ , we have  $x - \delta_I(t) = -\delta_I(t) + x$ , where  $+$  and  $-$  are counted in the group  $G$ .

We note that if  $M$  is an MV-algebra, then the concept of a lexicographic ideal coincides with a lexicographic ideal defined in [DFL], in addition, in such a case conditions (iv) and (v) are superfluous.

**Proposition 3.2.** Let  $(H, u)$  be a linearly ordered unital group and let  $G$  be an  $\ell$ -group. If we set  $I = \{(0, g) : g \in G^+\}$ , then  $I$  is a lexicographic ideal of  $M = \Gamma(H \overrightarrow{\times} G, (u, 0))$ .

*Proof.* It is clear that  $I$  is a normal ideal of  $M$  as well as it is prime because  $M/I \cong \Gamma(H, u)$  and the latter pseudo MV-algebra is linearly ordered.

We have  $x/I = 0/I$  iff  $x \in I$ . Assume  $(0, g)/I < (h, g')/I$ . Then  $(h, g) \notin I$  that yields  $h > 0$  and  $(0, g) < (h, g')$ . Hence, if  $x/I < y/I$ , then  $(y - x)/I > 0/I$  and  $y - x > 0$  and  $x < y$ .

Since  $M/I \cong \Gamma(H \overrightarrow{\times} \{0\}, (u, 0)) \subseteq \Gamma(H \overrightarrow{\times} G, (u, 0))$ , we see that  $I$  is retractive. We have  $I^- = I^\sim$ , so that  $\langle I \rangle = I \cup I^-$ .

Let  $h_1, h_2 \in [0, u]_H$  and  $g_1, g_2 \in G$  be such elements that  $(h_1, g_1), (h_2, g_2) \in \Gamma(H \overrightarrow{\times} G, (u, 0))$ . Then  $(h_i, g_i) \in \pi_I^{-1}(\{(h_i, 0)/I\})$  and  $\delta_i((h_i, 0)/I) = (h_i, 0)$ ,  $i = 1, 2$ . Hence,  $(h_1 + h_2, g_1 + g_2) - (h_1 + h_2, 0) = (0, g_1) + (0, g_2) = ((h_1, g_1) - (h_1, 0)) + ((h_2, g_2) - (h_2, 0))$  which proves that  $I$  is a lexicographic ideal.

Finally,  $(h, g) - (h, 0) = (0, g) = -(h, 0) + (h, g)$ , so that (v) holds.  $\square$

Let  $\text{LexId}(M)$  be the set of lexicographic ideals of  $M$ . Not every pseudo MV-algebra possesses a lexicographic ideal, e.g. the MV-algebra  $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 1))$  is such a case; it has a unique non-trivial ideal  $I = \{(0, n) : n \geq 0\}$ ,  $M/I \cong \Gamma(\frac{1}{2}\mathbb{Z}, 1)$  but  $M$  does not contain any copy of  $\Gamma(\frac{1}{2}\mathbb{Z}, 1)$ . On the other side, it can happen that a pseudo MV-algebra could have more lexicographic ideals as it is in the following example.

**Example 3.3.** We define MV-algebras:  $M_1 = \Gamma(\mathbb{Z} \overrightarrow{\times} (\mathbb{Z} \overrightarrow{\times} \mathbb{Z}), (1, (0, 0)))$ ,  $M_2 = \Gamma((\mathbb{Z} \overrightarrow{\times} \mathbb{Z}) \overrightarrow{\times} \mathbb{Z}, ((1, 0), 0))$ , and  $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (1, 0, 0))$  which are mutually isomorphic.

If we take the MV-algebra  $M$  from Example 3.3, we see that  $I_1 = \{(0, m, n) : m > 0, n \in \mathbb{Z} \text{ or } m = 0, n \geq 0\}$  and  $I_2 = \{(0, 0, n) : n \geq 0\}$  are only two lexicographic ideals of  $M$  and  $I_2 \subset I_1$ .

Similarly as in [Dvu6, Prop 7.1], we can show that if  $I$  and  $J$  are lexicographic ideals of  $M$ , then  $I \subseteq J$  or  $J \subseteq I$ .

**Definition 3.4.** Let  $(H, u)$  be a linearly ordered group. We say that a pseudo MV-algebra  $M$  is  $(H, u)$ -perfect, if there is a system  $(M_t : t \in [0, u]_H)$  of nonempty subsets of  $M$  such that it is an  $(H, u)$ -decomposition of  $M$ , i.e.  $M_s \cap M_t = \emptyset$  for  $s < t$ ,  $s, t \in [0, u]_H$  and  $\bigcup_{t \in [0, u]_H} M_t = M$ , and

- (a)  $M_s \leq M_t$  for all  $s < t$ ,  $s, t \in [0, u]_H$ ;
- (b)  $M_t^- = M_{u-t}$  and  $M_t^\sim = M_{-t+u}$  for each  $t \in [0, u]_H$ ;
- (c) if  $x \in M_v$  and  $y \in M_t$ , then  $x \oplus y \in M_{v \oplus t}$ , where  $v \oplus t = \min\{v + t, u\}$ .

We note that in view of property (a), we have that if  $x \in M_s$ ,  $y \in M_t$ , and  $s < t$ , then  $x < y$ . From (b), we have  $M_0^- = M_u = M_0^\sim$ . In addition, if  $M$  is symmetric, then in (b) we have  $M_t^- = M_{u-t} = M_t^\sim$  for each  $t \in [0, u]_H$ , and in [Dvu6], there was presented the notion of  $(H, u)$ -perfect pseudo MV-algebras only for symmetric pseudo MV-algebras.

For example, let us consider

$$M = \Gamma(H \overrightarrow{\times} G, (u, 0)). \quad (3.1)$$

We set  $M_0 = \{(0, g) : g \in G^+\}$ ,  $M_u := \{(u, -g) : g \in G^+\}$  and for  $t \in [0, u]_H \setminus \{0, u\}$ , we define  $M_t := \{(t, g) : g \in G\}$ . Then  $(M_t : t \in [0, u]_H)$  is an  $(H, u)$ -decomposition of  $M$  and  $M$  is an  $(H, u)$ -perfect pseudo MV-algebra.

Sometimes we will write also  $M = (M_t : t \in [0, u]_H)$  for an  $(H, u)$ -perfect pseudo MV-algebra.

**Theorem 3.5.** Let  $M = (M_t : t \in [0, u]_H)$  be an  $(H, u)$ -perfect pseudo MV-algebra.

- (i) Let  $a \in M_v$ ,  $b \in M_t$ . If  $v + t < u$ , then  $a + b$  is defined in  $M$  and  $a + b \in M_{v+t}$ ; if  $a + b$  is defined in  $M$ , then  $v + t \leq u$ . If  $a + b$  is defined in  $M$  and  $v + t = u$ , then  $a + b \in M_u$ .
- (ii)  $M_v + M_t$  is defined in  $M$  and  $M_v + M_t = M_{v+t}$  whenever  $v + t < u$ .
- (iii) If  $a \in M_v$  and  $b \in M_t$ , and  $v + t > u$ , then  $a + b$  is not defined in  $M$ .
- (iv) If  $a \in M_v$  and  $b \in M_t$ , then  $a \vee b \in M_{v \vee t}$  and  $a \wedge b \in M_{v \wedge t}$ .
- (v)  $M$  admits a state  $s$  such that  $M_0 \subseteq \text{Ker}(s)$ .
- (vi)  $M_0$  is a normal ideal of  $M$  such that  $M_0 + M_0 = M_0$  and  $M_0 \subseteq \text{Infin}(M)$ .
- (vii) The quotient pseudo MV-algebra  $M/M_0 \cong \Gamma(H, u)$ .
- (viii) Let  $M = (M'_t : t \in [0, u]_H)$  be another  $(H, u)$ -decomposition of  $M$  satisfying (a)–(c) of Definition 3.4, then  $M_t = M'_t$  for each  $t \in [0, u]_H$ .
- (ix)  $M_0$  is a prime ideal of  $M$ .

*Proof.* It is similar to the proof of [Dvu6, Thm 3.2] where it was assumed that  $M$  is symmetric, and therefore, we prove here only some items of them.

(ii) By (i), we have  $M_v + M_t \subseteq M_{v+t}$ . Suppose  $z \in M_{v+t}$ . Then, for any  $a \in M_v$ , we have  $a \leq z$ . Hence,  $b = -a + z = a/z = a^\sim \odot z$  is defined in  $M$  and  $b \in M_w$

for some  $w \in [0, u]_H$ . Since  $z = a + b \in M_{v+t} \cap M_{v+w}$ , we conclude  $t = w$  and  $M_{v+t} \subseteq M_v + M_t$ .

(iv) Inasmuch as  $a \wedge b = (a \oplus b^\sim) - b^\sim$ , we have by (c) of Definition 3.4,  $(a \oplus b^\sim) - b^\sim \in M_s$ , where  $s = ((v - t + u) \wedge u) - (-t + u) = v \wedge t$ . Using a de Morgan law, we have  $a \vee b \in M_{v \vee t}$ .  $\square$

We note that a  $(\mathbb{Z}, 1)$ -perfect pseudo MV-algebra is in [DDT] called *perfect*. Equivalently, a symmetric pseudo MV-algebra  $M$  is perfect iff every element  $x$  of  $M$  is either infinitesimal or co-infinitesimal.

In addition, a  $(\frac{1}{n}\mathbb{Z}, 1)$ -perfect pseudo MV-algebra is said to be  $n$ -perfect, see [Dvu4].

Now we present the main result of this section, a representation of a pseudo MV-algebra with a lexicographic ideal as a lexicographic pseudo MV-algebra.

For each  $\ell$ -group  $G$  and each unital linearly ordered group  $(H, u)$ , we define the pseudo MV-algebra

$$\mathcal{M}_{H,u}(G) := \Gamma(H \overrightarrow{\times} G, (u, 0)). \quad (3.2)$$

**Theorem 3.6.** *Let  $M$  be a pseudo MV-algebra and let  $I$  be a lexicographic ideal of  $M$ . Then there are a linearly ordered unital group  $(H, u)$  such that  $E/I \cong \Gamma(H, u)$  and an  $\ell$ -group  $G$  with  $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$  such that  $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$ .*

*In addition, if there is an  $\ell$ -group  $G'$  such that  $M \cong \Gamma(H \overrightarrow{\times} G', (u, 0))$ , then  $G'$  is isomorphic to  $G$ .*

*Proof.* According to a basic representation Theorem 2.1, we can assume that  $M = \Gamma(K, v)$  for some unital  $\ell$ -group  $(K, v)$ . Since  $I$  is lexicographic, then  $I$  is normal and prime, so that  $M/I$  is a linear pseudo MV-algebra. There is a linearly ordered unital group  $(H, u)$  such that  $M/I \cong \Gamma(H, u)$ ; without loss of generality, we assume  $M/I = \Gamma(H, u)$ .

Let  $\pi_I : M \rightarrow M/I$  be the canonical projection. For any  $t \in [0, u]_H$ , we set  $M_t := \pi_I^{-1}(\{t\})$ ; then  $M_0 = I$ . We assert that  $(M_t : t \in [0, u]_H)$  is an  $(H, u)$ -decomposition of  $M$ . Indeed, since  $\pi_I$  is surjective, every  $M_t$  is non-empty. The decomposition  $(M_t : t \in [0, u]_H)$  has the following properties: (a) Let  $x \in M_s$  and  $y \in M_t$  for  $s < t$ ,  $s, t \in [0, u]_H$ , then  $x < y$ . Indeed, since  $\pi_I(x) = s < t < \pi_I(y)$  and  $x < y$  because  $I$  is strict. (b)  $M_t^- = M_{u-t}$  and  $M_t^\sim = M_{-t+u}$  for each  $t \in [0, u]_H$  which holds because  $\pi_I$  is a homomorphism. (c)  $M_s \oplus M_t \subseteq M_{s \oplus t}$  for  $s, t \in [0, u]_H$ , where  $s \oplus t := \min\{s + t, u\}$ . Indeed,  $x \in M_s$  and  $y \in M_t$ , then  $\pi_I(x \oplus y) = \pi_I(x) \oplus \pi_I(y) = s \oplus t$ , so that  $M_s \oplus M_t \subseteq M_{s \oplus t}$ .

By (vi) of Theorem 3.5,  $M_0$  is an associative cancellative semigroup satisfying conditions of Birkhoff's Theorem [Bir, Thm XIV.2.1], [Fuc, Thm II.4], which guarantee that  $M_0$  is a positive cone of a unique (up to isomorphism) directed po-group  $G$ . Since  $M_0$  is a lattice, we have that  $G$  is an  $\ell$ -group. In addition, since  $\langle I \rangle = I \cup I^-$  is a perfect pseudo MV-algebra, we have by [DDT, Prop 5.2],  $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ .

For each  $t \in [0, u]_H$ , we set  $c_t = \delta_I(t)$ . Then (i)  $c_{s+t} = c_s + c_t$  if  $s + t \leq u$ , (ii)  $c_0 = 0$  and  $c_u = 1$ , and (iii)  $\{c_t\} = \delta_I(M/I) \cap M_t$ ,  $t \in [0, u]_H$ .

Take the  $(H, u)$ -perfect pseudo MV-algebra  $\mathcal{M}_{H,u}(G)$  defined by (3.2), and define a mapping  $\phi : M \rightarrow \mathcal{M}_{H,u}(G)$  by

$$\phi(x) := (t, x - c_t) \quad (3.3)$$



whenever  $x \in M_t$  for some  $t \in [0, u]_H$ , where  $x - c_t$  denotes the difference taken in the group  $K$ .

*Claim 1:  $\phi$  is a well-defined mapping.*

Indeed,  $M_0$  is in fact the positive cone of an  $\ell$ -group  $G$  which is a subgroup of  $K$ . Let  $x \in M_t$ . For the element  $x - c_t \in K$ , we define  $(x - c_t)^+ := (x - c_t) \vee 0 = (x \vee c_t) - c_t \in M_0$  (when we use (iii) of Theorem 3.5) and similarly  $(x - c_t)^- := -((x - c_t) \wedge 0) = c_t - (x \wedge c_t) \in M_0$ . This implies that  $x - c_t = (x - c_t)^+ - (x - c_t)^- \in G$ .

*Claim 2: The mapping  $\phi$  is an injective and surjective homomorphism of pseudo effect algebras.*

We have  $\phi(0) = (0, 0)$  and  $\phi(1) = (u, 0)$ . Let  $x \in M_t$ . Using (v) of Definition 3.1, we have  $x^- \in M_{u-t}$ ,  $\phi(x) = (t, x - c_t)$  and  $\phi(x^-) = (u - t, x^- - c_{u-t}) = (u - t, -c_{u-t} + x^-) = (u - t, c_t - 1 + 1 - x) = (u - t, c_t - x) = (u, 0) - (t, x - c_t) = \phi(x)^-$ . In an analogous way,  $\phi(x^\sim) = \phi(x)^\sim$ .

Now given  $x, y \in M$  and let  $x + y$  be defined in  $M$ . Then  $x \in M_{t_1}$  and  $y \in M_{t_2}$ . Since  $x \leq y^-$ , we have  $t_1 \leq u - t_2$  so that  $\phi(x) \leq \phi(y^-) = \phi(y)^-$  which means  $\phi(x) + \phi(y)$  is defined in  $\mathcal{M}_{H,u}(G)$ . Using (iv) of Definition 3.1, we conclude  $\phi(x + y) = (t_1 + t_2, x + y - c_{t_1+t_2}) = (t_1 + t_2, x + y - (c_{t_1} + c_{t_2})) = (t_1, x - c_{t_1}) + (t_2, y - c_{t_2}) = \phi(x) + \phi(y)$ .

Assume  $\phi(x) \leq \phi(y)$  for some  $x \in M_t$  and  $y \in M_v$ . Then  $(t, x - c_t) \leq (v, y - c_v)$ . If  $t = v$ , then  $x - c_t \leq y - c_t$  so that  $x \leq y$ . If  $t < v$ , then  $x \in M_t$  and  $y \in M_v$  so that  $x < y$ . Therefore,  $\phi$  is injective.

To prove that  $\phi$  is surjective, assume two cases: (i) Take  $g \in G^+ = M_0$ . Then  $\phi(g) = (0, g)$ . In addition  $g^- \in M_u$  so that  $\phi(g^-) = \phi(g)^- = (0, g)^- = (u, 0) - (0, g) = (u, -g)$ . (ii) Let  $g \in G$  and  $t$  with  $0 < t < u$  be given. Then  $g = g_1 - g_2$ , where  $g_1, g_2 \in G^+ = M_0$ . Since  $c_t \in M_t$ , we have  $g_2 \leq c_t$ , so that  $-g_2 + c_t = g_2/c_t$  exists in  $M$  and it belongs to  $M_t$ , which yields  $g + c_t = g_1 + (-g_2 + c_t) \in M_t$ . Hence,  $\phi(g + c_t) = (t, g)$  when we have used the property (iv) of Definition 3.1.

*Claim 3: If  $x \leq y$ , then  $\phi(y \setminus x) = \phi(y) \setminus \phi(x)$  and  $\phi(x / y) = \phi(x) / \phi(y)$ . In particular,  $\phi(x^-) = \phi(x)^-$  and  $\phi(x^\sim) = \phi(x)^\sim$  for each  $x \in M$ .*

It follows from the fact that  $\phi$  is a homomorphism of pseudo effect algebras.

*Claim 4:  $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$  and  $\phi(x \vee y) = \phi(x) \vee \phi(y)$ .*

We have,  $\phi(x), \phi(y) \geq \phi(x \wedge y)$ . If  $\phi(x), \phi(y) \geq \phi(w)$  for some  $w \in M$ , we have  $x, y \geq w$  and  $x \wedge y \geq w$ . In the same way we deal with  $\vee$ .

*Claim 5:  $\phi$  is a homomorphism of pseudo MV-algebras.*

It is necessary to show that  $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$ . This follows straightforwardly from the previous claims and equality (2.2).

Consequently,  $M$  is isomorphic to  $\mathcal{M}_{H,u}(G)$  as pseudo MV-algebras.

If  $M \cong \Gamma(H \overrightarrow{\times} G', (u, 0))$  for some  $\ell$ -group  $G'$ , let  $\psi : \Gamma(H \overrightarrow{\times} G, (u, 0)) \rightarrow \Gamma(H \overrightarrow{\times} G', (u, 0))$  be an isomorphism of pseudo MV-algebras. Then  $\psi$  induces another  $(H, u)$ -decomposition  $(M'_t : t \in [0, u]_H)$  of  $\Gamma(H \overrightarrow{\times} G', (u, 0))$ , where  $M'_t = \psi(M_t)$ . By Theorem 3.5(viii), we see that  $\psi(\{(0, g) : g \in G^+\}) = \{(0, g') : g' \in G'^+\}$  which proves that  $G$  and  $G'$  are isomorphic  $\ell$ -groups.  $\square$

We note that in Example 3.3, the pseudo MV-algebra has two lexicographic ideals  $I_1$  and  $I_2$ , so that it has two representations as lexicographic pseudo MV-algebras, namely one as  $M_1$  and the second as  $M_2$ . One is  $(\mathbb{Z}, 1)$ -perfect and the second is  $(\mathbb{Z} \xrightarrow{\times} \mathbb{Z}, (1, 0))$ -perfect, and of course the linear unital groups  $(H_1, u_1) := (\mathbb{Z}, 1)$  and  $(H_2, u_2) := (\mathbb{Z} \xrightarrow{\times} \mathbb{Z}, (1, 0))$  are not isomorphic.

We say that a pseudo MV-algebra  $M$  is *I-representable* if  $I$  is a lexicographic ideal of  $M$  and  $M \cong \Gamma(H \xrightarrow{\times} G, (u, 0))$ , where  $(H, u)$  is a linearly ordered unital group such that  $M/I \cong \Gamma(H, u)$  and  $G$  is an  $\ell$ -group such that  $\langle I \rangle \cong \Gamma(\mathbb{Z} \xrightarrow{\times} G, (1, 0))$ ; the existences of  $(H, u)$  and  $G$  are guaranteed by Theorem 3.6. Since the linearly ordered unital group  $(H, u)$  is uniquely (up to isomorphism of unital  $\ell$ -groups) determined by the retractive ideal  $I$ , we can say also that  $M$  is also *(H, u)-lexicographic*. This notion is well defined because if there is another lexicographic ideal  $J$  of  $M$  such that  $M/J \cong \Gamma(H, u)$ , then  $(\pi_I^{-1}(\{t\}) : t \in [0, u]_H)$  and  $(\pi_J^{-1}(\{t\}) : t \in [0, u]_H)$  are two  $(H, u)$ -decompositions of  $M$ , so that by Theorem 3.5(viii), they are the same, in particular  $I = \pi_I^{-1}(\{0\}) = \pi_J^{-1}(\{0\}) = J$ .

Now we define another notion that is very closely connected with lexicographic pseudo MV-algebras.

**Definition 3.7.** We say that an  $(H, u)$ -perfect pseudo MV-algebra  $M = \Gamma(K, v)$  is *strongly (H, u)-perfect* if it is  $(H, u)$ -perfect, and if there is a system of elements  $(c_t : t \in [0, u]_H)$  of  $M$  such that

- (i)  $c_t \in M_t$  for each  $t \in [0, u]_H$ ;
- (ii)  $c_{s+t} = c_s + c_t$  if  $s + t \leq u$ ;
- (ii)  $c_u = 1$ ;
- (iv)  $(x + y) - c_{s+t} = (x - c_s) - (y - c_t)$  if  $x \in M_s, y \in M_t, s + t \leq u$ , where  $+$  and  $-$  are counted in the  $\ell$ -group  $K$ ;
- (v) for each  $t \in [0, u]_H$  and each  $x \in M_t$ , we have  $x - c_t = -c_t + x$ , where  $+$  and  $-$  are counted in the  $\ell$ -group  $K$ .

In view of (ii), we have  $c_0 + c_0 = c_0$ , so that  $c_0 = 0$ .

Now we show that, for any pseudo MV-algebra  $M$ , *I-representability* and strong  $(H, u)$ -perfectness are equivalent.

**Theorem 3.8.** *Let  $M$  be a pseudo MV-algebra and  $(H, u)$  be a linearly ordered group. The following assertions are equivalent:*

- (i)  $M$  is *I-representable* and  $M/I = \Gamma(H, u)$ .
- (ii)  $M$  is *strongly (H, u)-perfect*.
- (iii)  $M$  is *(H, u)-lexicographic*.

*Proof.* (i)  $\Rightarrow$  (ii). Let  $I$  be a lexicographic ideal of  $M$ . From the proof of Theorem 3.6, we see that if we put  $M_t := \pi_I^{-1}(\{t\})$  for each  $t \in [0, u]$ , then the system  $(M_t : t \in [0, u]_H)$  is an  $(H, u)$ -decomposition of  $M$ . The system  $(c_t : t \in [0, u]_H)$ , where  $c_t := \delta_I(t)$  for  $t \in [0, u]_H$ , proves  $M$  is strongly  $(H, u)$ -perfect.

(ii)  $\Rightarrow$  (i). Let  $(H, u)$  be a linearly ordered unital group and let  $(M_t : t \in [0, u]_H)$  be an  $(H, u)$ -decomposition and let  $(c_t : t \in [0, u]_H)$  be a system of elements of  $M$  satisfying conditions (i)–(v) of Definition 3.7. By Theorem 3.5(vi),(ix),(vii),  $I := M_0$  is a normal and prime ideal of  $M$  such that  $M/I \cong \Gamma(H, u)$ ; without loss of generality, we assume  $M/I = \Gamma(H, u)$ . (Indeed, if  $\iota_I : M/I \rightarrow \Gamma(H, u)$  is an isomorphism, there is a unique isomorphism  $\delta'_I : \Gamma(H, u) \rightarrow M_I := \delta'_I([0, u]_H)$  such that  $\delta_I = \delta'_I \circ \iota_I$ , and then we deal with  $\pi'_I := \iota_I \circ \pi_I, \delta'_I$  and  $id_{[0, u]_H}$  instead of  $\pi_I$ ,

$\delta_I$  and  $id_{M/I}$ , respectively.) In addition, the system  $(M_t : t \in [0, u]_H)$  is a unique. Let  $\pi_I : M \rightarrow M/I$  be the canonical homomorphism. Then  $x \sim_I y$  iff there is  $t \in [0, u]_H$  such that  $x, y \in M_t$ . Hence,  $\pi_I^{-1}(\{t\}) = M_t$  for each  $t \in [0, u]_H$ . In view of (a) of Definition 3.4, we see that  $I$  is a strict ideal of  $M$ .

Let  $M' := \{c_t : t \in [0, u]_H\}$ . In view of Definition 3.7, we see that  $M'$  is a pseudo effect algebra such that  $c_t^- = c_{u-t}$  and  $c_t^\sim = c_{-t+u}$  for each  $t \in [0, u]_H$ . This pseudo effect algebra is linearly ordered via  $c_s < c_t$  iff  $s < t$  so that  $c_{s \wedge t} = c_s \wedge c_t$ , where  $\wedge$  is taken in the pseudo MV-algebra  $M$ . In addition, the order taken in  $M$  and the one taken in the pseudo effect algebra  $M'$  coincide in  $M'$ , so that  $c_{t-s} = c_t - c_s$  and  $c_{-s+t} = -c_s + c_t$  if  $t > s$ . In view of (2.2), we conclude, that  $M'$  is a subalgebra of the pseudo MV-algebra  $M$ , and  $M' = M_I$ . If we define  $\delta_I : \Gamma(H, u) \rightarrow M'$  via  $\delta_I(t) = c_t$ , we see that  $\delta_I$  is a homomorphism such that  $\pi_I \circ \delta_I = id_{M/I}$  which proves that  $I$  is a retractive ideal. In view of (iv) of Definition 3.7, we see that  $I$  is a lexicographic ideal of  $M$ .

(i)  $\Leftrightarrow$  (iii). It is evident.  $\square$

**Corollary 3.9.** *Let  $M = (M_t : t \in [0, u]_H)$  be a strongly  $(H, u)$ -perfect pseudo MV-algebra with a fixed system of elements  $(c_t : t \in [0, u]_H)$  satisfying conditions of Definition 3.7. Then  $I := M_0$  is a lexicographic ideal of  $M$  such that  $M/I \cong \Gamma(H, u)$ ,  $M' = \{c_t : t \in [0, u]_H\}$  is a subalgebra of  $M$  isomorphic to  $\Gamma(H, u)$ , and the mapping  $\delta_I : M/I \rightarrow M'$  given by  $\delta_I(t) = c_t$ ,  $t \in [0, u]_H$ , is an isomorphism such that  $\pi_I \circ \delta_I = id_{M/I}$ .*

*Proof.* It follows from the proof of implication (ii)  $\Rightarrow$  (i) of Theorem 3.8.  $\square$

As an additional corollary, we have that every strongly  $(H, u)$ -perfect pseudo MV-algebra is  $(H, u)$ -lexicographic, and consequently, it is lexicographic.

**Corollary 3.10.** *Let  $(H, u)$  be a linearly ordered unital group. If  $M$  is a strongly  $(H, u)$ -perfect pseudo MV-algebra, then there is a unique up to isomorphism  $\ell$ -group  $G$  such that*

$$M \cong \Gamma(H \overrightarrow{\times} G, (u, 0)).$$

*Proof.* It follows from Theorem 3.8 and Theorem 3.6.  $\square$

#### 4. CATEGORICAL EQUIVALENCE

As we have seen,  $I$ -representable pseudo MV-algebras, strongly  $(H, u)$ -perfect pseudo MV-algebras, and  $(H, u)$ -lexicographic pseudo MV-algebras, where  $I$  is a lexicographic ideal of  $M$  such that  $M/I \cong \Gamma(H, u)$ , are the same objects.

In this section, we establish the categorical equivalence of the category of  $(H, u)$ -lexicographic pseudo MV-algebras with the variety of  $\ell$ -groups. This extends the categorical representation perfect MV-algebras proved in [DiLe1], the one of strongly  $n$ -perfect pseudo MV-algebras from [Dvu4], the one of  $\mathbb{H}$ -perfect pseudo MV-algebras from [Dvu5] as well as the categorical equivalence of lexicographic MV-algebras from [DFL] with the variety of  $\ell$ -groups.

Thus we assume that  $(H, u)$  is in this section a fixed linearly ordered unital group.

Let  $M$  be an  $(H, u)$ -lexicographic pseudo MV-algebra, i.e.  $M$  is a pseudo MV-algebra with a lexicographic ideal  $I$  such that  $M/I \cong \Gamma(H, u)$ , and in addition, there is a subalgebra  $M_I$  of  $M$  which is isomorphic to  $M/I$  and there is an isomorphism  $\delta_I : M/I \rightarrow M_I$  such that  $\pi_I \circ \delta_I = id_{M/I}$ . Without loss of generalization, we will

assume that  $M/I = \Gamma(H, u)$ ; otherwise, we change  $\pi_I$ ,  $\delta_I$  and  $id_{M/I}$  to  $\pi'_I := \iota_I \circ \pi_I$ ,  $\delta'_I = \delta_I \circ \iota_I^{-1}$  (an isomorphism from  $\Gamma(H, u)$  onto  $M_I$ ), and  $id_{\Gamma(H, u)}$ , respectively, where  $\iota_I : M/I \rightarrow \Gamma(H, u)$  is an isomorphism.

In other words, our  $(H, u)$ -lexicographic pseudo MV-algebra can be characterized by a quadruplet  $(M, I, M_I, \delta_I)$ , and  $\delta_I$  will be now an isomorphism from  $\Gamma(H, u)$  onto  $M_I$ .

For example, let  $M = \Gamma(H \overrightarrow{\times} G, (u, 0))$  for some  $\ell$ -group  $G$ . If we put  $J = \{(0, g) : g \in G^+\}$ , then  $\Gamma(H \overrightarrow{\times} G, (u, 0))/J \cong \Gamma(H, u)$ ,  $M_J := \{(t, 0) : t \in [0, u]_H\}$  is a subalgebra of  $\Gamma(H \overrightarrow{\times} G, (u, 0))$  isomorphic to  $\Gamma(H, u) \cong \Gamma(H \overrightarrow{\times} G, (u, 0))/J$  with an isomorphism  $\delta_J(t) = (t, 0)$ ,  $t \in [0, u]_H$ , satisfying  $\pi_J \circ \delta_J = id_{M/J}$ .

Therefore, we define the category  $\mathcal{LP}_s\mathcal{MV}_{H, u}$  of  $(H, u)$ -lexicographic pseudo MV-algebras whose objects are quadruplets  $(M, I, M_I, \delta_I)$ , where  $I$  is a lexicographic ideal of  $M$  such that  $M/I \cong \Gamma(H, u)$ ,  $M_I$  is a subalgebra of  $M$  isomorphic to  $M/I$  with an isomorphism  $\delta_I : M/I \rightarrow M_I$  such that  $\pi_I \circ \delta_I = id_{M/I}$ . For example  $(\Gamma(H \overrightarrow{\times} G, (u, 0)), J, M_J, \delta_J)$  is an object of  $\mathcal{LP}_s\mathcal{MV}_{H, u}$ .

If  $(M_1, I_1, M_{I_1}, \delta_{I_1})$  and  $(M_2, I_2, M_{I_2}, \delta_{I_2})$  are two objects of  $\mathcal{LP}_s\mathcal{MV}_{H, u}$ , then a morphism  $f : (M_1, I_1, M_{I_1}, \delta_{I_1}) \rightarrow (M_2, I_2, M_{I_2}, \delta_{I_2})$  is a homomorphism of pseudo MV-algebras  $f : M_1 \rightarrow M_2$  such that

$$f(I_1) \subseteq I_2, \quad f(M_{I_1}) \subseteq M_{I_2}, \quad \text{and} \quad f \circ \delta_{I_1} = \delta_{I_2}.$$

It is straightforward to verify that  $\mathcal{LP}_s\mathcal{MV}_{H, u}$  is a well-defined category.

Now let  $\mathcal{LG}$  be the category whose objects are  $\ell$ -groups and morphisms are homomorphisms of  $\ell$ -groups.

Define a mapping  $\mathcal{M}_{H, u}^s : \mathcal{LG} \rightarrow \mathcal{LP}_s\mathcal{MV}_{H, u}$  as follows: for  $G \in \mathcal{LG}$ , let

$$\mathcal{M}_{H, u}^s(G) := (\Gamma(H \overrightarrow{\times} G, (u, 0)), J, H_J, \delta_J) \quad (4.1)$$

and if  $h : G \rightarrow G_1$  is an  $\ell$ -group homomorphism, then

$$\mathcal{M}_{H, u}^s(h)(t, g) = (t, h(g)), \quad (t, g) \in \Gamma(H \overrightarrow{\times} G, (u, 0)). \quad (4.2)$$

**Proposition 4.1.**  $\mathcal{M}_{H, u}^s$  is a well-defined functor that is a faithful and full functor from the category  $\mathcal{LG}$  of  $\ell$ -groups into the category  $\mathcal{LP}_s\mathcal{MV}_{H, u}$  of  $(H, u)$ -lexicographic pseudo MV-algebras.

*Proof.* First we show that  $\mathcal{M}_{H, u}^s$  is a well-defined functor. Alias, we have to show that if  $h$  is a morphism of  $\ell$ -groups, then  $\mathcal{M}_{H, u}^s(h)$  is a morphism in the category  $\mathcal{LP}_s\mathcal{MV}_{H, u}$ . Let  $\mathcal{M}_{H, u}^s(G) := (\Gamma(H \overrightarrow{\times} G, (u, 0)), J, H_J, \delta_J)$  and  $\mathcal{M}_{H, u}^s(G_1) := (\Gamma(H \overrightarrow{\times} G_1, (u, 0)), J_1, H_{J_1}, \delta_{J_1})$ . Then  $\mathcal{M}_{H, u}^s(h)(J) \subseteq J_1$  and  $\mathcal{M}_{H, u}^s(h)(H_J) \subseteq H_{J_1}$ . Check, let  $t \in \Gamma(H, u)$ , then  $\mathcal{M}_{H, u}^s(h) \circ \delta_J(t) = \mathcal{M}_{H, u}^s(h)(t, 0) = (t, 0) = \delta_{J_1}(t)$ .

Let  $h_1$  and  $h_2$  be two morphisms from  $G$  into  $G'$  such that  $\mathcal{M}_{H, u}^s(h_1) = \mathcal{M}_{H, u}^s(h_2)$ . Then  $(0, h_1(g)) = (0, h_2(g))$  for each  $g \in G^+$ , consequently  $h_1 = h_2$ .

To prove that  $\mathcal{M}_{H, u}^s$  is a full functor, let  $f : (\Gamma(H \overrightarrow{\times} G, (u, 0)), J, H_J, \delta_J) \rightarrow (\Gamma(H \overrightarrow{\times} G', (u, 0)), J', H_{J'}, \delta_{J'})$  be a morphism from  $\mathcal{LP}_s\mathcal{MV}_{H, u}$ . We claim that  $f(t, 0) = (t, 0)$  for each  $t \in \Gamma(H, u)$ . Indeed, we have  $f(t, 0) = f(\delta_J(t)) = \delta_{J'}(t) = (t, 0)$ .

In addition, we have  $f(0, g) = (0, g')$  for a unique  $g' \in G'^+$ . Define a mapping  $h : G^+ \rightarrow G'^+$  by  $h(g) = g'$  iff  $f(0, g) = (0, g')$ . Then  $h(g_1 + g_2) = h(g_1) + h(g_2)$  if  $g_1, g_2 \in G^+$ . Assume now that  $g \in G$  is arbitrary. Then  $g = g^+ - g^-$ , where

$g^+ = g \vee 0$  and  $g^- = -(g \wedge 0)$ , and  $g = -g^- + g^+$ . If  $g = g_1 - g_2$ , where  $g_1, g_2 \in G^+$ , then  $g^+ + g_2 = g^- + g_1$  and  $h(g^+) + h(g_2) = h(g^-) + h(g_1)$  which shows that  $h(g) = h(g_1) - h(g_2)$  is a well-defined extension of  $h$  from  $G^+$  onto  $G$ .

If  $0 \leq g_1 \leq g_2$ , then  $(0, g_1) \leq (0, g_2)$ , which means that  $h$  is a mapping preserving the partial order.

We have yet to show that  $h$  preserves  $\wedge$  in  $G$ , i.e.,  $h(a \wedge b) = h(a) \wedge h(b)$  whenever  $a, b \in G$ . Let  $a = a^+ - a^-$  and  $b = b^+ - b^-$ , and  $a = -a^- + a^+$ ,  $b = -b^- + b^+$ . Since  $h((a^+ + b^-) \wedge (a^- + b^+)) = h(a^+ + b^-) \wedge h(a^- + b^+)$ . Subtracting  $h(b^-)$  from the right hand and  $h(a^-)$  from the left hand, we obtain the statement in question.

By Theorem 2.1, the homomorphism  $f : \Gamma(H \overrightarrow{\times} G, (u, 0)) \rightarrow \Gamma(H \overrightarrow{\times} G', (u, 0))$  can be uniquely extended to a morphism of unital  $\ell$ -groups  $\bar{f} : (H \overrightarrow{\times} G, (u, 0)) \rightarrow (H \overrightarrow{\times} G', (u, 0))$ . Then  $f(t, g) = \bar{f}(t, g) = \bar{f}(t, 0) + \bar{f}(0, g) = (t, 0) + (0, h(g)) = (t, h(g))$ .

Finally, we have proved that  $h$  is a homomorphism of  $\ell$ -groups, and  $\mathcal{M}_{H,u}^s(h) = f$  as claimed.  $\square$

We note that by a *universal group* for a pseudo MV-algebra  $M$  we mean a pair  $(G, \gamma)$  consisting of an  $\ell$ -group  $G$  and a  $G$ -valued measure  $\gamma : M \rightarrow G^+$  (i.e.,  $\gamma(a + b) = \gamma(a) + \gamma(b)$  whenever  $a + b$  is defined in  $M$ ) such that the following conditions hold: (i)  $\gamma(M)$  generates  $G$ . (ii) If  $K$  is a group and  $\phi : M \rightarrow K$  is an  $K$ -valued measure, then there is a group homomorphism  $\phi^* : G \rightarrow K$  such that  $\phi = \phi^* \circ \gamma$ .

Due to [Dvu2], every pseudo MV-algebra admits a universal group, which is unique up to isomorphism, and  $\phi^*$  is unique. The universal group for  $M = \Gamma(G, u)$  is  $(G, id)$  where  $id$  is an embedding of  $M$  into  $G$ .

**Proposition 4.2.** *The functor  $\mathcal{M}_{H,u}^s$  from the category  $\mathcal{LG}$  into  $\mathcal{LP}_s\mathcal{MV}_{H,u}$  has a left-adjoint.*

*Proof.* The proof follows the ideas of the proof of [Dvu6, Prop 8.3], but we present it in its fullness to be self-contained.

Every object  $(M, I, M_I, \delta_I)$  in  $\mathcal{LP}_s\mathcal{MV}_{H,u}$ , has a universal arrow  $(G, f)$ , i.e.,  $G$  is an object in  $\mathcal{LG}$  and  $f$  is a homomorphism from the pseudo MV-algebra  $M$  into  $\mathcal{M}_{H,u}^s(G)$  such that if  $G'$  is an object from  $\mathcal{LG}$  and  $f'$  is a homomorphism from  $M$  into  $\mathcal{M}_{H,u}^s(G')$ , then there exists a unique morphism  $f^* : G \rightarrow G'$  such that  $\mathcal{M}_{H,u}^s(f^*) \circ f = f'$ .

Since by Theorem 3.8,  $M$  is also a strongly  $(H, u)$ -perfect pseudo MV-algebra with an  $(H, u)$ -decomposition  $(M_t : t \in [0, u]_H)$  and with a family  $(c_t : t \in [0, u]_H)$  of elements of  $M$  satisfying Definition 3.7, by Theorem 3.6, there is a unique (up to isomorphism of  $\ell$ -groups)  $\ell$ -group  $G$  such that  $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$ . By [Dvu2, Thm 5.3],  $(H \overrightarrow{\times} G, \gamma)$  is a universal group for  $M$ , where  $\gamma : M \rightarrow \Gamma(H \overrightarrow{\times} G, (u, 0))$  is defined by  $\gamma(a) = (t, a - c_t)$  if  $a \in M_t$ , see (3.3).

We assert that  $(G, \gamma)$  is a universal arrow for  $(M, I, M_I, \delta_I)$ . Clearly we have that  $\gamma : (M, I, M_I, \delta_I) \rightarrow (\Gamma(H \overrightarrow{\times} G, (u, 0)), J, H_J, \delta_J)$  is a morphism. In addition, assume that  $f' : (M, I, M_I, \delta_I) \rightarrow (\Gamma(H \overrightarrow{\times} G', (u, 0)), J', H_{J'}, \delta_{J'})$  is an arbitrary morphism. There is a unique homomorphism of unital  $\ell$ -groups  $\alpha : (H \overrightarrow{\times} G, (u, 0)) \rightarrow (H \overrightarrow{\times} G', (u, 0))$  such that  $\Gamma(\alpha) \circ \gamma = f'$ , where  $\Gamma$  is a functor from the category of unital  $\ell$ -groups into the category of pseudo MV-algebras, see Theorem 2.1. Since  $f'$  and  $\gamma$  are morphisms of the category  $\mathcal{LP}_s\mathcal{MV}_{H,u}$ , then  $\gamma \circ \psi = \psi_J$  and

$f' \circ \delta_I = \delta_{J'}$  which entail  $\Gamma(\alpha)(J) = \Gamma(\alpha) \circ \gamma \circ \gamma^{-1}(J) = f' \circ (\gamma^{-1}(J)) = f'(I) \subseteq J'$ ,  $\Gamma(\alpha)(H_j) = \Gamma(\alpha) \circ \gamma \circ \gamma^{-1}(H_j) = f'(M_I) \subseteq J'$ , and  $\Gamma(\alpha) \circ \delta_J(t) = \Gamma(\alpha) \circ \gamma \circ \delta_I(t) = f' \circ \delta_I(t) = \psi_{J'}(t)$ ,  $t \in \Gamma(H, u)$ . This proves that  $\Gamma(\alpha)$  is a morphism from  $(\Gamma(H \overrightarrow{\times} G, (u, 0)), J, H_J, \delta_J)$  into  $(\Gamma(H \overrightarrow{\times} G', (u, 0)), J', H_{J'}, \delta_{J'})$ .

Then for each  $g \in G$ , there is a unique  $g' \in G'^+$  such that  $\Gamma(\alpha)(0, g) = (0, g')$ . Hence, there is an  $\ell$ -group homomorphism  $\beta : G \rightarrow G'$  such that  $\Gamma(\alpha)(0, g) = (0, \beta(g))$  for each  $g \in G$ . This gives  $\mathcal{M}_{H,u}^s(\beta)(h, g) = (h, \beta(g)) = (h, 0) + (0, \beta(g)) = \alpha(h, 0) + \alpha(0, g) = \alpha(h, g)$  and  $\mathcal{M}_{H,u}^s(\beta) \circ \gamma = f'$ , consequently,  $(G, \gamma)$  is a universal arrow for  $(M, I, M_I, \delta_I)$ .  $\square$

Define a mapping  $\mathcal{P}_{H,u}^s : \mathcal{LP}_s\mathcal{MV}_{H,u} \rightarrow \mathcal{LG}$  via  $\mathcal{P}_{H,u}^s(M, I, M_I, \delta_I) := G$  whenever  $(H \overrightarrow{\times} G, f)$  is a universal group for  $M$ . It is clear that if  $f_0$  is a morphism from  $(M, I, M_I, \delta_I) \in \mathcal{LP}_s\mathcal{MV}_{H,u}$  into another one  $(N, I_N, N_{I_N}, \delta_N)$ , then  $f_0$  can be uniquely extended to an  $\ell$ -group homomorphism  $\mathcal{P}_{H,u}^s(f_0)$  from  $G$  into  $G_1$ , where  $(H \overrightarrow{\times} G_1, f_1)$  is a universal group for an  $(H, u)$ -lexicographic pseudo MV-algebra  $N$ . Therefore, we have the following statement.

**Proposition 4.3.** *The mapping  $\mathcal{P}_{H,u}^s$  is a functor from the category  $\mathcal{LP}_s\mathcal{MV}_{H,u}$  into the category  $\mathcal{LG}$  which is a left-adjoint of the functor  $\mathcal{M}_{H,u}^s$ .*

Now we present the basic result of this section on a categorical equivalence of the category of  $(H, u)$ -lexicographic pseudo MV-algebras and the category of  $\mathcal{LG}$ .

**Theorem 4.4.** *The functor  $\mathcal{M}_{H,u}^s$  defines a categorical equivalence of the category  $\mathcal{LG}$  and the category  $\mathcal{LP}_s\mathcal{MV}_{H,u}$  of  $(H, u)$ -lexicographic pseudo MV-algebras.*

*In addition, suppose that  $h : \mathcal{M}_{H,u}^s(G) \rightarrow \mathcal{M}_{H,u}^s(G')$  is a homomorphism of pseudo MV-algebras, then there is a unique homomorphism  $f : G \rightarrow G'$  of  $\ell$ -groups such that  $h = \mathcal{M}_{H,u}^s(f)$ , and*

- (i) *if  $h$  is surjective, so is  $f$ ;*
- (ii) *if  $h$  is injective, so is  $f$ .*

*Proof.* In view of [MaL, Thm IV.4.1], it is necessary to show that, for any  $(H, u)$ -lexicographic pseudo MV-algebra  $(M, I, M_I, \delta_I)$ , there is an object  $G$  in  $\mathcal{LG}$  such that  $\mathcal{M}_{H,u}^s(G)$  is isomorphic to  $(M, I, M_I, \delta_I)$ . To establish that, we take a universal arrow  $(H \overrightarrow{\times} G, f)$  of  $M$ . Then  $\mathcal{M}_{H,u}^s(G)$  and  $(M, I, M_I, \delta_I)$  are isomorphic.  $\square$

## 5. WEAKLY LEXICOGRAPHIC PSEUDO MV-ALGEBRAS

In this section, we generalize the notion of a retractive ideal, a lexicographic ideal, and a lexicographic pseudo MV-algebra in order to characterize pseudo MV-algebras that can be represented in the form  $\Gamma(H \overrightarrow{\times} G, (u, b))$ , where  $b \in G^+$  is not necessarily the zero element.

First we introduce another notion of  $(H, u)$ -perfect pseudo MV-algebra.

**Definition 5.1.** We say that an  $(H, u)$ -perfect pseudo MV-algebra  $M = \Gamma(K, v)$  is *weakly  $(H, u)$ -perfect* if it is  $(H, u)$ -perfect, and if there is a system of elements  $(c_t : t \in [0, u]_H)$  of  $M$  such that

- (i)  $c_t \in M_t$  for each  $t \in [0, u]_H$ ;
- (ii)  $c_{s+t} = c_s + c_t$  if  $s + t \leq u$ ;
- (iii)  $(x + y) - c_{s+t} = (x - c_s) - (y - c_t)$  if  $x \in M_s, y \in M_t, s + t \leq u$ , where  $+$  and  $-$  are counted in the  $\ell$ -group  $K$ ;

- (iv) for each  $t \in [0, u]_H$  and each  $x \in M_t$ , we have  $x - c_t = -c_t + x$ , where  $+$  and  $-$  are counted in the  $\ell$ -group  $K$ .

For example, let  $M = \Gamma(H \overrightarrow{\times} G, (u, b))$  for some  $b \in G^+$ . We set  $c_t = (t, 0)$ ,  $t \in [0, u]_H$ , then  $M$  is weakly  $(H, u)$ -perfect.

In view of (ii), we have  $c_0 + c_0 = c_0$ , so that  $c_0 = 0$ . Comparing with Definition 3.7, we see that we do not assume that  $c_u = 1$ , therefore, the subset  $\{c_t : t \in [0, u]_H\}$  is not necessarily a subalgebra of  $M$ . Of course every strongly  $(H, u)$ -perfect pseudo MV-algebra is a weakly  $(H, u)$ -perfect one, but as we show below, the converse is not true in general.

We note that according to Theorem 3.6 and Theorem 3.8, every strongly  $(H, u)$ -perfect pseudo MV-algebra  $M$  is of the form  $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$  and it admits a lexicographic ideal  $I$  such that  $M/I \cong \Gamma(H, u)$ . The MV-algebra  $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 1))$  is weakly  $(\mathbb{Z}, 2)$ -perfect; it contains elements  $c_0 = (0, 0)$ ,  $c_1 = (1, 0)$ ,  $c_2 = (2, 0)$ , however, as it was already mentioned, it has a unique non-trivial ideal  $I = \{(0, n) : n \geq 0\}$  and it is not lexicographic because  $M$  does not contain any copy of  $\Gamma(\frac{1}{2}\mathbb{Z}, 1)$ .

On the other hand, it can happen, that a weakly  $(H, u)$ -perfect pseudo MV-algebra is also strongly  $(H, u)$ -perfect. Indeed, MV-algebras  $M_1 = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 2))$  and  $M_2 = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, -2))$  are isomorphic to  $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 0))$ ; we define isomorphisms  $\theta_i : M \rightarrow M_i$ ,  $i = 1, 2$ , such that  $\theta_1(0, n) = (0, n)$ ,  $\theta_1(1, n) = (1, n+1)$ ,  $\theta_1(2, n) = (2, n+2)$  and  $\theta_2(0, n) = (0, n)$ ,  $\theta_2(1, n) = (1, n-1)$ ,  $\theta_2(2, n) = (2, n-2)$ . In  $M_1$  we can define two families  $c_1 = (0, 0)$ ,  $c_2 = (1, 0)$ ,  $c_3 = (2, 0)$  which shows that  $M_1$  is weakly  $(\mathbb{Z}, 2)$ -perfect, and the family  $c'_1 = (0, 0)$ ,  $c'_2 = (1, 1)$ ,  $c'_3 = (2, 2)$  which shows that it is also strongly  $(\mathbb{Z}, 2)$ -perfect. The first family does not form a subalgebra of  $M$ , and the second one does form. For the MV-algebra  $M_2$  we have two analogous families:  $c_1 = (0, 0)$ ,  $c_2 = (1, -2)$ ,  $c_3 = (2, -4)$  and  $c'_1 = (0, 0)$ ,  $c'_2 = (1, -1)$ ,  $c'_3 = (2, -2)$ .

Now we introduce a weaker form of a retractive ideal.

A normal ideal  $I$  of a pseudo MV-algebra  $M$  is said to be *weakly retractive* if the canonical projection  $\pi_I : M \rightarrow M/I$  is weakly retractive, i.e. there is a mapping  $\delta_I : M/I \rightarrow M$  such that (i)  $\pi_I \circ \delta_I = id_{M/I}$ , (ii)  $\delta_I(x/I + y/I) = \delta_I(x/I) + \delta_I(y/I)$  whenever  $x/I + y/I \leq 1/I$ , where  $+$  is the partial addition induced by  $\oplus$  in pseudo MV-algebras. Then (1)  $\delta_I(0/I) = 0$ , (2)  $\delta_I(x/I) < 1$  whenever  $x/I < 1/I$ , (3)  $\delta_I$  is injective.

We note that a weakly retractive ideal is retractive whenever  $\delta_I(1/I) = 1$ .

In addition, we define a weakly lexicographic ideal:

**Definition 5.2.** A normal ideal  $I$  of a pseudo MV-algebra  $M = \Gamma(G, u)$ ,  $\{0\} \neq I \neq M$ , is said to be *weakly lexicographic* if

- (i)  $I$  is strict;
- (ii)  $I$  is weakly retractive;
- (iii)  $I$  is prime;
- (iv) for each  $s, t \in [0, u]_H$ , where  $\Gamma(H, u) := M/I$ , such that  $s + t \leq u$  and for each  $x \in \pi_I^{-1}(\{s\})$  and  $y \in \pi_I^{-1}(\{t\})$ , we have  $x + y - \delta_I(s + t) = (x - \delta_I(s)) + (y - \delta_I(t))$ , where  $+$  and  $-$  are counted in the group  $G$ ,
- (v) for each  $t \in [0, u]_H$  and each  $x \in \pi_I^{-1}(\{t\})$ , we have  $x - \delta_I(t) = -\delta_I(t) + x$ , where  $+$  and  $-$  are counted in the group  $G$ .

For example,  $M = \Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 1))$  has no lexicographic ideal, but  $I = \{(0, n) : n \geq 0\}$  is a weakly lexicographic ideal of  $M$ .

If  $M = \Gamma(H \overrightarrow{\times} G, (u, b))$  with  $b \in G^+$ , then  $I = \{(0, g) : g \in G^+\}$  is a weakly lexicographic ideal of  $M$ , and  $(M_t : t \in [0, u]_H)$ , where  $M_t = \{(t, g) : (t, g) \in M\}$ , is an  $(H, u)$ -decomposition of  $M$ . In particular,  $I^- = I^\sim$ .

Now we characterize  $(H, u)$ -perfect pseudo MV-algebras that can be represented in the form  $\Gamma(H \overrightarrow{\times} G, (u, b))$  for some  $b \in G^+$ .

**Theorem 5.3.** *Let  $M$  be a pseudo MV-algebra and let  $I$  be a weakly lexicographic ideal of  $M$ . Then there are a linearly ordered unital group  $(H, u)$  such that  $E/I \cong \Gamma(H, u)$ , an  $\ell$ -group  $G$  with  $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$  and an element  $b \in G^+$  such that  $M \cong \Gamma(H \overrightarrow{\times} G, (u, b))$ .*

*In addition, if there is an  $\ell$ -group  $G'$  such that  $M \cong \Gamma(H \overrightarrow{\times} G', (u, b'))$  where  $b' \in G'^+$ , then  $G'$  is isomorphic to  $G$ .*

*Proof.* We follow the main steps of the proof of Theorem 3.6. Thus let  $M = \Gamma(K, v)$  for some unital  $\ell$ -group  $(K, v)$ ,  $I$  be a weakly lexicographic ideal of  $M$ , and let  $\pi_I : M \rightarrow M/I$  be the canonical projection. Since  $I$  is prime, there is a linearly ordered unital group  $(H, u)$  such that  $M/I \cong \Gamma(H, u)$ ; without loss of generality, we assume  $M/I = \Gamma(H, u)$ . Then  $(M_t : t \in [0, u]_H)$  is an  $(H, u)$ -decomposition of  $M$ , where  $M_t := \pi_I^{-1}(\{t\})$ ,  $t \in [0, u]_H$ .

Being  $M_0 = I$ ,  $M_0$  is an associative cancellative semigroup,  $M_0$  is a positive cone of an  $\ell$ -group  $G$ . By [DDT, Prop 5.2],  $\langle I \rangle \cong \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ .

Given  $t \in [0, u]_H$  we put  $c_t = \delta_I(t)$ . Since  $I$  is weakly lexicographic,  $M$  with  $(c_t : t \in [0, u]_H)$  is weakly  $(H, u)$ -perfect,  $\{c_t\} = M_t \cap \delta_I(M/I)$ . Then  $\delta_I(u) \leq 1$  and we put  $b = 1 - \delta_I(u) \in G^+$ , where  $-$  is subtraction counted in the  $\ell$ -group  $G \subseteq K$ . We note that according to (v) of Definition 5.2,  $1 - c_u = -c_u + 1$ .

*Claim:*  $c_t + b = c_t + b$  for every  $t \in [0, u]_H$ .

Indeed, according to (iv) of Definition 5.2, we have  $b = 1 - c_u = x + x^\sim - c_{t+(-t+u)} = x - c_t + x^\sim - c_{-t+u} = -c_t + x - x + 1 - (-c_t + c_u) = -c_t + 1 - c_u + c_t = -c_t + b + c_t = b$ , so that  $b + c_t = c_t + b$ .

We define a pseudo MV-algebra

$$\mathcal{M}_{H,u}(G, b) := \Gamma(H \overrightarrow{\times} G, (u, b)),$$

and define a mapping  $\phi : M \rightarrow \mathcal{M}_{H,u}(G, b)$  by

$$\phi(x) := (t, x - c_t) \tag{5.1}$$

whenever  $x \in M_t$  for some  $t \in [0, u]_H$ , where  $x - c_t$  is a difference taken in the group  $K$ .

Similarly as in the proof of Theorem 3.6,  $\phi$  is a well-defined mapping such that (i)  $\phi(0) = (0, 0)$ ,  $\phi(1) = (u, 1 - c_u) = (1, b)$ . (ii) If  $x \in M_t$ , then  $x^\sim \in M_{-t+u}$ ,  $x^- \in M_{u-t}$ , so that  $\phi(x) = (t, x - c_t)$  and  $\phi(x^\sim) = (-t + u, (-x + 1) - (c_{-t+u})) = (-t + u, -x + b + c_t)$ . On the other side,  $\phi(x)^\sim = -(t, x - c_t) + (u, b) = (-t + u, c_t - x + b) = \phi(x^\sim)$  when we have used (v) of Definition 5.2 and Claim. In a similar way we have,  $\phi(x^-) = (u - t, x^- - c_{u-t}) = (u - t, -c_{u-t} + 1 - x) = (u - t, c_t - c_u + 1 - x) = (u - t, b + c_t - x) = (u, b) - (t, x - c_t) = \phi(x)^-$ .



Using the ideas of the proof of Theorem 3.6 and (iv) of Definition 5.2, we have  $\phi(x + y) = \phi(x) + \phi(y)$ ,  $\phi$  is injective and surjective which proves that  $\phi$  is an isomorphism of pseudo effect algebras, consequently, using (2.2),  $\phi$  is an isomorphism of the pseudo MV-algebras  $M$  and  $\mathcal{M}_{H,u}(G, b)$ .

Finally, if  $M \cong \Gamma(H \overrightarrow{\times} G', (u, b'))$  for some  $G'$  and  $b' \in G'^+$ , then in a similar manner as at the end of the proof of Theorem 3.6, we can prove that  $G$  and  $G'$  are isomorphic  $\ell$ -groups.  $\square$

It is worthy to recall that in Theorem 5.3, if  $\Gamma(H \overrightarrow{\times} G, (u, b))$  is isomorphic to  $\Gamma(H \overrightarrow{\times} G', (u, b'))$ , where  $b \in G^+$  and  $b' \in G'^+$ , then  $G$  and  $G'$  are isomorphic  $\ell$ -groups, but  $b$  does not map necessarily to  $b'$ . Indeed, take  $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 0))$  and  $\Gamma(\mathbb{Z} \overrightarrow{\times} \mathbb{Z}, (2, 2))$ . It was already proved that they are isomorphic, but  $b = 0$  does not map to  $b = 2$  under any isomorphism from  $\mathbb{Z}$  onto itself.

Using ideas from the proof of Theorem 3.8, it is possible to show that a pseudo MV-algebra is weakly  $(H, u)$ -perfect iff  $M$  has a weakly lexicographic ideal  $I$  such that  $M/I \cong \Gamma(H, u)$ . Hence, such pseudo MV-algebras can be called also *weakly  $(H, u)$ -lexicographic pseudo MV-algebras*.

Finally we present a categorical equivalence of the category of weakly  $(H, u)$ -lexicographic pseudo MV-algebras with the category of pointed  $\ell$ -groups in an analogous way as it was done in the previous section.

Let  $M$  be a weakly  $(H, u)$ -lexicographic pseudo MV-algebra. It has a weakly lexicographic ideal  $I$  such that  $M/I \cong \Gamma(H, u)$ , and there is an injective mapping  $\delta_I : \Gamma(H, u) \cong M/I \rightarrow M$  satisfying the conditions of Definition 5.2; we set  $M_I := \delta_I(M/I)$ .

We define the category  $\mathcal{WLP}_s\mathcal{MV}_{H,u}$  of weakly  $(H, u)$ -lexicographic pseudo MV-algebras whose objects are quadruplets  $(M, I, M_I, \delta_I)$ , where  $M$  is a weakly  $(H, u)$ -perfect pseudo MV-algebra with a weakly lexicographic ideal  $I$  such that  $M/I \cong \Gamma(H, u)$ ,  $\delta_I : \Gamma(H, u) \cong M/I \rightarrow M_I$ , see Definition 5.2. If  $M = \Gamma(H \overrightarrow{\times} G, (u, b))$ , we set  $J = \{(0, g) : g \in G^+\}$ ,  $M_J = \{(t, 0) : t \in [0, u]_H\}$ ,  $\delta_J(t) = (t, 0)$ ,  $t \in [0, u]_H$ .

If  $(M_1, I_1, M_{I_1}, \delta_{I_1})$  and  $(M_2, I_2, M_{I_2}, \delta_{I_2})$  are two objects of  $\mathcal{WLP}_s\mathcal{MV}_{H,u}$ , then a morphism  $f : (M_1, I_1, M_{I_1}, \delta_{I_1}) \rightarrow (M_2, I_2, M_{I_2}, \delta_{I_2})$  is a homomorphism of pseudo MV-algebras  $f : M_1 \rightarrow M_2$  such that

$$f(I_1) \subseteq I_2, \quad f(M_{I_1}) \subseteq M_{I_2}, \quad \text{and} \quad f \circ \delta_{I_1} = \delta_{I_2}.$$

Now we define the category of pointed  $\ell$ -groups,  $\mathcal{PLG}$ , i.e., the objects are couples  $(G, b)$ , where  $G$  is an  $\ell$ -group and  $b \in G^+$  is a fixed element, and morphisms are homomorphisms of  $\ell$ -groups preserving fixed elements. We note that the class of pointed  $\ell$ -groups is a variety whereas the class of unital  $\ell$ -groups not.

Define a mapping  $\mathcal{M}_{H,u}^w : \mathcal{PLG} \rightarrow \mathcal{LP}_s\mathcal{MV}_{H,u}$  as follows: for  $(G, b) \in \mathcal{PLG}$ , let

$$\mathcal{M}_{H,u}^w(G, b) := (\Gamma(H \overrightarrow{\times} G, (u, b)), J, H_J, \delta_J)$$

and if  $h : (G, b) \rightarrow (G_1, b_1)$  is an  $\ell$ -group homomorphism preserving fixed elements, then

$$\mathcal{M}_{H,u}^w(h)(t, g) = (t, h(g)), \quad (t, g) \in \Gamma(H \overrightarrow{\times} G, (u, b)).$$

Finally, we present a categorical equivalence in the same way as it was proved in the previous section.

**Theorem 5.4.** *The functor  $\mathcal{M}_{H,u}^w$  defines a categorical equivalence of the category  $\mathcal{PLG}$  and the category  $\mathcal{WLP}_s\mathcal{MV}_{H,u}$  of weakly  $(H, u)$ -lexicographic pseudo MV-algebras.*

**Problem 5.5.** We note that the class of  $(H, u)$ -perfect pseudo MV-algebras does not form a variety. However, it would be interesting to know an equational basis of the variety of pseudo MV-algebras generated by the class of  $(H, u)$ -lexicographic (or weakly  $(H, u)$ -lexicographic) pseudo MV-algebras. For example, if  $(H, u) = (\mathbb{Z}, 1)$ , the basis is  $2.x^2 = (2.x)^2$ , see [DDT, Rem 5.6].

## 6. CONCLUSION

We have exhibited conditions when a pseudo MV-algebra  $M$  can be represented as an interval in the lexicographic product of a fixed linearly ordered unital group  $(H, u)$  with an  $\ell$ -group  $G$ , both groups are not necessarily Abelian. A crucial condition was the existence of a lexicographic normal ideal  $I$  such that  $M/I \cong \Gamma(H, u)$ , or equivalently,  $M$  is a strongly  $(H, u)$ -perfect pseudo MV-algebra (i.e.  $M$  can be decomposed into a system of comparable slices indexed by elements of the interval  $[0, u]_H$ ). Such algebras have a representation  $M \cong \Gamma(H \overrightarrow{\times} G, (u, 0))$ , we called them also  $(H, u)$ -lexicographic pseudo MV-algebras, Theorem 3.6 and Theorem 3.8, and they are not semisimple. We have shown that the category of  $(H, u)$ -lexicographic pseudo MV-algebras is categorically equivalent to the category of  $\ell$ -groups, Theorem 4.4.

In addition, we have also studied conditions when a pseudo MV-algebra can be represented in the form  $\Gamma(H \overrightarrow{\times} G, (u, b))$ , where  $b \in G^+$  is not necessarily the zero element. We call such algebras weakly  $(H, u)$ -lexicographic, or weakly  $(H, u)$ -perfect pseudo MV-algebras. A fundamental notion was a weak lexicographic ideal. Their representation is given in Theorem 5.3. This category is categorically equivalent to the category of pointed  $\ell$ -groups whose objects are pairs  $(G, b)$  where  $b \in G^+$  is a fixed element, Theorem 5.4.

We hope that this research will inspired an additional study of MV-algebras and pseudo MV-algebras that are not semisimple.

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